Random Graphs
Exercise Sheet 6

Question 1. Suppose we flip n biased coins, each which land heads with probability p and tails with probability $1-p$, independently of the others. Let X be the number of heads flipped, let Z_i be the result of the ith coin flip and consider the martingale given by $X_i = \mathbb{E}(X|\sigma(Z_1, \ldots, Z_i))$. Using the Azuma-Hoeffding inequality bound, for any $t \geq 0$, the probability

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t).$$

Compare this to the Chernoff bounds.

Question 2. Let $A \subseteq \{0,1\}^n$ have size $\epsilon 2^n$ and let λ be such that $\exp\left(-\frac{\lambda^2}{2}\right) = \epsilon$. Show that all but $\epsilon 2^n$ points in $\{0,1\}^n$ are at Hamming distance at most $2\lambda\sqrt{n}$ from A.

(Hint: The Hamming distance is a 1-Lipschitz function on $\{0,1\}^n$)

Question 3. Let p be fixed, $\varepsilon > 0$, $b = \frac{1}{1-p}$ and let $k = (2-\varepsilon)\log_b n$. Let Y be the largest size of a collection of 'edge-disjoint' independent sets of size k in $G_{n,p}$ and let $\mathcal{K}$ be the collection of all independent sets of size k in $G_{n,p}$. By choosing a random subset of $\mathcal{K}$ and using the alteration method, show that

$$\mathbb{E}(Y) \geq (1+o(1))p(1-p)\frac{n^2}{k^4}.$$

Deduce that

$$\mathbb{P}(\alpha(G_{n,p}) < k) \leq e^{-\tilde{\Omega}(n^2)},$$

where $\tilde{\Omega}$ means up to polylog factors.

Question 4. Suppose we generate a configuration F on a set W of $2m$ points via the procedure from the lecture - We sequentially choose an unmatched vertex x (according to some arbitrary rule, which might depend on the history of the procedure) and then choose an unmatched vertex y uniformly at random and add the pair $\{x, y\}$ to F. Show that this produces a uniformly random configuration.

Let $H \in \mathcal{G}_{n,\overline{d}}$ have exactly r parallel edges and t loops. Determine the probability that $G^*(n, \overline{d}) = H$.

Question 5. Given $k \in \mathbb{N}$ the random k-out graph is generated by choosing independently and uniformly for each vertex $v \in [n]$ a set of k neighbours. Show that for $k \geq 3$ this graph is whp connected.

(* What about for $k \leq 2$?)

Question 6. Let r be sufficiently large. Suppose we pick r perfect matchings $M_1, M_2, \ldots, M_r$ on the vertex set $[n]$ and consider the r-regular (multi-)graph G whose edge set is given by $\bigcup M_i$.

Show that there exists a constant α_r such that whp $e_G(S, [n] \setminus S) \geq \alpha_r |S|$ for every subset $S \subseteq [n]$ of size $|S| \leq \frac{n}{2}$.